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Approximate Truth and Łukasiewicz Logic

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Łukasiewicz' infinite-valued logic L has been used by a number of authors to study reasoning with vague, "fuzzy", or uncertain statements ([5], [2], [11], [12]). Scott and, more recently, Katz have also argued that L is appropriate for the logic of degrees of error or degrees of approximation to the truth ([6], p. 421 and [4], p. 773). The aim of this note is to show that L is not appropriate for this purpose, regardless of its possible application to vagueness, and to derive a few facts about alternative systems which are appropriate, or more nearly so.

Both Scott and Katz indicate that a leading idea in their analysis of degree of error is the case of equations $r = s$, where some metric is used to measure the distance between r and s. If r and s are reals, then a convenient example is the metric $|r - s|$. In this case, $r = s$ true iff $|r - s| = 0$, and if $r \neq s$, $|r - s|$ measures the degree of error of the equation $r = s$, larger values indicating larger errors.

Scott and Katz both use L to extend to compound formulas the measure of error that a metric like $|r - s|$ gives for equations. My contention that L does not do this correctly is based on the following claim: If we are to take at all seriously the idea that the "truth values" of our system measure the degree of error, we must insist that true statements have zero error. This aim is, of course, already met for the equation example just mentioned, but we ought to meet it for all statements whose "nearness" to the truth is to be assessed.

In fact, it is just the property that true statements are zero distance from the truth—call it the "accuracy property"—that distinguishes an assessment of accuracy, at which Scott and Katz are clearly aiming, from an assessment of proximity to the *comprehensive* truth, which is the aim (or an aim) of the theories of so-called verisimilitude or truthlikeness and of theories of vagueness. A statement can be inaccurate only by stating something which is false. It may be less than the comprehensive truth, on the other hand, merely by sins of omission. Similarly, a statement might possess some degree of vagueness and still be true, but it could not possess any degree of inaccuracy and still be true.[1]

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Let us apply the accuracy property to a multivalued logic designed to measure error. For simplicity, we consider the language whose only predicate is '=', with constants 'r', 's', 't', etc., and variables 'x', 'y', 'z', etc. The symbols '$\vee$', '&', '$\supset$', '$\sim$', '(x)', and '$(\exists x)$', stand for disjunction, conjunction, material conditional, negation, universal and existential quantification, respectively. A structure M for this language consists of a nonempty universe U, denotations for the terms 'r', 's', 't', etc., and a function F: $U \times U \rightarrow [0,1]$ giving a metric on U with values in the interval $[0,1]$. We also use a denotation function $d(u,\sigma)$ defined as follows: let the function σ

$$\sigma: \{`x', `y', `z', \ldots\} \rightarrow U$$

be any assignment of values in U to each variable 'x', 'y', etc. Then $d(u,\sigma)$ is the denotation of u if u is a constant, or the value assigned to u by σ if u is a variable. We are clearly assuming here that the value 0 represents the truth, and that values in $(0,1]$ represent various degrees of error or inaccuracy, with 1 the maximum possible error. We let $\|p\|$ be the degree of error assigned by F to the statement p. If we interchange 0 with 1 in the usual Łukasiewicz continuum-valued system, we get definition clauses equivalent to those adopted by Katz for $\|p\|(\sigma)$, which is $\|p\|$ considered as a function of the sequence σ:

(1)
$$\begin{aligned}
\|u = v\|(\sigma) &= F(d(u,\sigma),\, d(v,\sigma)) \\
\|{\sim}p\|(\sigma &= 1 - \|p\|(\sigma) \\
\|p \vee q\|(\sigma) &= \min(\|p\|(\sigma),\, \|q\|(\sigma)) \\
\|p \,\&\, q\|(\sigma) &= \max(\|p\|(\sigma),\, \|q\|(\sigma)) \\
\|p \supset q\|(\sigma) &= \max(0, \|q\|(\sigma) - \|p\|(\sigma)) \\
\|(x)p\|(\sigma) &= \sup_{\sigma'} \|p\|(\sigma'),
\end{aligned}$$

where the supremum is taken over those σ' which agree with σ except possibly on the assignment to 'x'.

$$\|(\exists x)p\|(\sigma) = \inf_{\sigma'} \|p\|(\sigma'),$$

where the infinum is taken over all σ' which agree with σ except possibly on the value assigned to 'x'.

Where σ is understood, we can omit the '(σ)' in '$\| \;\; \|(\sigma)$', but we also officially define

$$\|p\| = \sup_{\sigma} \|p\|(\sigma),$$

the supremum of $\|p\|(\sigma)$ over all assignments σ. If $0 < \|p\| < 1$, we say that p is *intermediate*. We also restate the accuracy property that true statements have zero error as:

(2) If p is true then $\|p\| = 0$.

That definition (1) does not conform to this condition (2) may be seen from the following proposition:

Proposition 1
(a) *If p is intermediate, then* $\|p \vee {\sim}p\| > 0$.
(b) *If p and q are intermediate and* $\|q\| > \|p\|$ *then* $\|p \supset q\| > 0$.

Proof: Immediate from definition (1).

It follows from this proposition that (2) is violated in several ways by the Łukasiewicz clauses in (1). In (a), $p \vee \sim p$ is obviously true under the usual interpretation of the connectives since it is a tautology.[2] If $\|p\| = \frac{1}{2}$, however, then $\|p \vee \sim p\| = \frac{1}{2}$, which seems a long way from the truth on a scale of 0 to 1. In (b), p and q must both be false since their $\|\ \|$ values are positive (using (2)), so that $p \supset q$ must be true although its $\|\ \|$ value is positive. If $\|p\| = \frac{1}{3}$, for example, and $\|q\| = \frac{2}{3}$, then $\|p \supset q\| = \frac{1}{3}$, although $p \supset q$ is true.

A little more analysis of these two cases shows that the '$\sim$' and '$\supset$' clauses can be modified to avoid violating the accuracy principle (2). These modifications can be carried out and still retain the convenient feature that L shares with most multivalued logics, namely truth-functionality. That is, if '$\Box$' stands for any of the binary connectives of L, there is function $f_\Box$: $[0,1]x[0,1] \rightarrow [0,1]$ such that $\|p \Box q\| = f_\Box(\|p\|, \|q\|)$, and a similar function for negation.

Using (2) and truth-functionality as joint requirements, we give a heuristic "derivation" of an alternative multivalued system S, beginning with negation: If $\|p\| > 0$, then p must be false, so $\sim p$ will be true, and $\|\sim p\| = 0$. If $\|p\| = 0$, however, p could conceivably be false, since (2) only requires that true p have $\|p\| = 0$, not that false p must have $\|p\|$ different from zero. It is convenient to suppose this, however, and by doing so, we would only be requiring of all p what the metric F already guarantees for equations. This supposition does not, however, determine a unique value for $\|\sim p\|$ when $\|p\| = 0$. $\|\sim p\|$ must have some positive value, and it is convenient to choose $\|\sim p\| = 1$. Thus we obtain the following revised definition for $\|\sim p\|$:

$$(3) \quad \|\sim p\| = \begin{cases} 0 \text{ if } \|p\| > 0 \\ 1 \text{ if } \|p\| = 0. \end{cases}$$

For the revised definition of $\|p \supset q\|$, condition (2) requires a value of 0 whenever $\|p\| > 0$, since $p \supset q$ must be true in that case. For $\|p\| = 0$, $\|p \supset q\|$ must be a function of $\|q\|$ if $\supset$ is to be truth-functional. It also seems clear that in this case, $\|p \supset q\|$ should be an increasing function of $\|q\|$. That is, if p is true and q is false, then $\|p \supset q\|$ should be farther from the truth (i.e., from 0) the "more false" q is. Since the simplest increasing function of $\|q\|$ is $\|q\|$ itself, we arrive at the following clause for $\supset$:

$$(4) \quad \|p \supset q\| = \begin{cases} 0 \text{ if } \|p\| > 0 \\ \|q\| \text{ if } \|p\| = 0. \end{cases}$$

For comparison with other multivalued systems, it is convenient to let x stand for any element of [0,1] other than the end points 0 and 1, and "compress" the continuum-valued function $\|\ \|$ into a three-valued logic given by the following tables:

(5)

$\supset$	0	x	1
0	0	x	1
x	0	0	0
1	0	0	0

$\vee$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

&	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

$\sim$	
0	1
x	0
1	0

The multivalued system S based on these tables clearly satisfies (2), but has an important drawback, that logically equivalent statements need not have the same S-values. For example, if $\|p\| = x$, $0 < x < 1$, then $\|{\sim}p\| = 0$, and $\|{\sim}{\sim}p\| = 1$, so $\|p\| \neq \|{\sim}{\sim}p\|$. This unhappy property is not, however, a result of some mistake in the formulation of S, as the following proposition shows:

Proposition 2 *If* $\| \;\; \|$ *assigns degrees of error in* $[0,1]$ *to statements p by a truth-functional multivalued logic, if p and q are intermediate with* $\|p\| \neq \|q\|$, *and for all* p', $\|p'\| = 0$ *if* p' *is true (the accuracy principle), then* $\|p\| \neq \|{\sim}{\sim}p\|$ *or* $\|q\| \neq \|{\sim}{\sim}q\|$.

Proof: Since p and q are intermediate, they must be false, since we would have $\|p\| = 0$ or $\|q\| = 0$ if either were true. Since p and q are false, ${\sim}p$ and ${\sim}q$ are true, and by the accuracy principle, $\|{\sim}p\| = \|{\sim}q\| = 0$. Since negation is truth functional, $\|{\sim}{\sim}p\| = \|{\sim}{\sim}q\|$. Since $\|p\| \neq \|q\|$ by hypothesis, we have either $\|p\| \neq \|{\sim}{\sim}p\|$ or $\|q\| \neq \|{\sim}{\sim}q\|$.

The consequences of this proposition are sweeping. No logic of accuracy worthy of the name is possible if there are not at least three degrees of accuracy. At least two are necessary merely to distinguish between true and false statements, and at least three if any distinctions among degrees of falsity are to be made. The accuracy principle is unavoidable as well, since without it some true statement will be counted as inaccurate. It follows from Proposition 2 either that the logic of approximate truth is not truth-functional, and *a fortiori*, not Łukasiewicz logic, or that some logically equivalent statements are assigned different degrees of inaccuracy. This suggests that one should look into non-truth-functional alternatives. Hilpinen ([3]), for example, proposes a logic based on Brouwerian modal logic. I will discuss this alternative further in a future paper, but a few important points remain to explore about S.

Although there is very little in the literature on systems based on the truth tables (5), Slupecki ([7]) investigated a related system based on $\supset$ and $\sim$ as in the tables (5), plus a third unary operation R which has the table:

R	
0	x
x	0
1	1

Slupecki gives an axiomatization and completeness proof for this system, but his only designated value is 0, so he only considers how truth, and not approximate truth, is preserved by inferences.

This is a critical limitation, however, because a major reason for studying approximate truth is to discover which inferences from *nearly* true statements have *nearly* true conclusions. Some familiar valid arguments can permit arbitrarily large errors in conclusions, even for small errors in the premises. For example, consider the *modus ponens* argument:

$$(6) \quad \frac{\begin{array}{c} p \supset q \\ p \end{array}}{q}.$$

If $\|p\| = .01$ or any other positive distance from the truth—no matter how small—then $p \supset q$ is true, so the accuracy principle (2) implies $\|p \supset q\| = 0$. Thus both premises can be as close to the truth as we like, without imposing any conditions on the size of $\|q\|$. Note that this discontinuity in *modus ponens* is not just a property of the system S, but follows directly from (2). Additional conditions can be obtained to guarantee that sufficiently small errors in the premises will yield small errors in the conclusion—i.e., a kind of continuity for arguments—but the above example shows that *modus ponens* will not, in general, be continuous.

It is often possible to compensate for inaccuracy in one premise of a valid argument by strengthening another premise. For example, if we assume that the second premise of (6) is precisely true, then we will have $\|q\| \leq \alpha$ for the conclusion q if $\|p \supset q\| \leq \alpha$, i.e., if the error in the first premise is at most α. On the other hand, if conditions stronger than truth are imposed on the first premise, the error in the conclusion will be at most α if the error in the second premise is at most α. (Sufficient conditions for continuity of arguments like *modus ponens* were announced in [9].) Katz in [4] and Aronson, et al., in [1] investigate inferences which preserve small or large truth values within their respective Łukasiewicz-based systems, but if the argument of this paper is correct, that work will not apply to the reservation of accuracy without substantial modifications. There remain the two possibilities of the system S and of non-truth-functional logics to be investigated further.

NOTES

1. For further discussion of this criterion, see [3] and [10].
2. Here, as throughout this paper, I attempt to develop a notion of approximate truth suited to a classical interpretation of the logical connectives. There are, of course, interpretations of the propositional calculus in which $p \vee \sim p$ need not be true, but this will not happen in the classical interpretation of '$\vee$', which is the subject of this paper.

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